Quadratic centers defining Elliptic Surfaces

Sébastien GAUTIER

February 1, 2008

Abstract

Let X be a quadratic vector field with a center whose generic orbits are algebraic curves of genus one. To each X we associate an elliptic surface (a smooth complex compact surface which is a genus one fibration). We give the list of all such vector fields and determine the corresponding elliptic surfaces.

1 Introduction

The second part of the 16th Hilbert problem asks for an upper bound to the number of limit cycles of a plane polynomial vector field of degree less or equal to n. Even in the case of quadratic systems ($n = 2$) the problem remains open. An infinitesimal version of the 16th Hilbert problem can be formulated as follows:

Find an upper bound $Z(f, n)$ to the number of limit cycles of a polynomial vector field of degree n, close to a polynomial vector field with a first integral f.

The associated foliation on the plane is defined by

$$R^{-1}df + \varepsilon\omega = 0 \tag{1}$$

where $R^{-1}df = Pdx + Qdy$ is a given polynomial one-form, $\deg P, \deg Q \leq n$, R^{-1} is an integrating factor, and ω is a polynomial one-form of degree n with coefficients depending analytically on the small parameter ε.

A progress in solving the infinitesimal 16th Hilbert problem is achieved mainly in the case when f is a polynomial of degree three, or $F = y^2 + P(x)$ where P is a polynomial of degree four see [I02, P90, G01]. A key point is that the generic leaves $\Gamma_c = \{f = c\} \subset \mathbb{C}^2$ of the polynomial foliation $R^{-1}df = 0$ are elliptic curves. We expect that the perturbations of more general polynomial foliations with elliptic leaves (which we call "elliptic foliations") can be studied along the same lines. This leads naturally to the following (open) problem.

For a given $n > 1$ determine, up to an affine equivalence, the elliptic polynomial foliations $Pdx + Qdy = 0$, $\deg P, \deg Q \leq n$.

The present paper adresses the above problem in the quadratic case, $n = 2$. In view of applications to the 16th Hilbert problem most important is the case when the non-perturbed foliation is real and possesses a center. Such foliations are well-known since Dulac (1908) and Kapteyn (1912). Moreover, when the leaves of the foliation (the orbits of the quadratic vector field) are algebraic curves, there is a (rational) first integral f [J79]. Reminding the classification of quadratic vector fields with a center, a rational first integral of the foliation induced is thus of four different kinds:

$$f = P_3(x, y) \text{ with } P_3 \in \mathbb{R}_3[x, y] \quad \text{(Hamiltonian case)} \tag{2}$$

$$f = x^\lambda(y^2 + P_2(x)) \text{ with } \begin{cases} \lambda \in \mathbb{Q} \\ P_2 \in \mathbb{R}_2[x, y] \end{cases} \quad \text{(reversible case)} \tag{3}$$

$$f = x^\lambda y^\mu(ax + by + c) \text{ with } \begin{cases} \lambda, \mu \in \mathbb{Q} \\ P_2 \in \mathbb{R}_2[x, y] \\ \text{a,b,c real numbers} \end{cases} \quad \text{(Lotka-Volterra case)} \tag{4}$$

$$f = P_2(x,y)^{-3}P_3(x,y)^2 \text{ with } \left\{ \begin{array}{l} P_3 \in \mathbb{R}_3[x,y] \\ P_2 \in \mathbb{R}_2[x,y] \end{array} \right. \quad \text{(codimension 4 case)} \tag{5}$$

In section 2 we give the classification, up to an affine equivalence, of all elliptic foliations with a first integral of the form (3) or (4). The Hamiltonian cases obviously induce an elliptic foliation and have already been studied. Remarks concerning the codimension 4 case can be found in [GGI, G07]. In our classification, the base field is supposed to be $\mathbb{C}$, so all parameters $a, b, c, \lambda, ...$ are complex. We get a finite list of such foliations with a center as well several infinite series of degenerated foliations which can not have a center (when the base field is $\mathbb{R}$). Most of these elliptic foliations were not previously studied in the context of the 16th Hilbert problem (but see [CLLL06, YL02, ILY05]).

The section 3 deals with the topology of the singular surface induced. An elliptic foliation in $\mathbb{C}^2$ (or more generally a foliation with an algebraic first integral f) gives rise canonically to an elliptic surface as follows. Suppose that f is chosen in such a way that the generic fiber of the map $f : \mathbb{C}^2 \to \mathbb{C}$ is an irreducible algebraic curve. The induced rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ have a finite number of points of indetermination. After a finite number of blow-ups of $\mathbb{P}^2$ at these points we get (by the Hironaka's desingularisation theorem) an induced analytic map

$$\mathbb{P}^2 \subset K \xrightarrow{f} \mathbb{P}^1$$

where K is a smooth complex surface. We may further suppose that K is minimal in the sense that the fibers do not contain exceptional curves of first kind. The pair (K, f) is then the elliptic surface associated to the elliptic foliation $R^{-1}df = 0$. It is unique up to a fiber preserving isomorphism. In this last section, we compute the singular fibers of the elliptic surfaces obtained. The singular fibers of an elliptic surface are classified by Kodaira [Ko63]. Such computations are 2-folds. First of all, it permits to identifiate isomorphic elliptic surfaces of non affine equivalent foliations, wich on its own is of interest. But the most important is that it immediately gives the local monodromy of the singular fibers. Here, the number of singular fibers (except in the Hamiltonian case) do not exceed 4 so the local monodromy of the singular fibers gives a good description of the (global) monodromy group of the associated Picard-Fuchs equation (or equivalently, the homological invariant of the surface [Ko63]), which on its turn is necessary when studying zeros of Abelian integrals (or limit cycles of the perturbed foliation (1)), see [G01, P90, I02, GGI, G07] for details.

2 Quadratic centers which define elliptic foliations

Let $\mathcal{F} = \mathcal{F}(\omega)$ be a foliation on the plane $\mathbb{C}^2$ defined by a differential form $\omega = Pdx + Qdy$. We say that $\mathcal{F}(df)$ is *elliptic* provided that its generic leaves are elliptic curves. As stated in the Introduction, in the present paper we suppose that $\mathcal{F}$ has, eventually after an affine change of the variables in $\mathbb{C}^2$, a first integral of the form (3) or (4). Such a foliation will be called *reversible* (having an integral of the form (3) but not (4)), of *Lotka-Voltera type* (having an integral of the form (4) but not (3)), or of reversible Lotka-Voltera type.

2.1 The reversible case

An elliptic foliation of Lotka-Voltera type has three invariant lines. From this we deduce that a reversible Lotka-Voltera foliation has always a first integral $f = x^\lambda(y^2 + P_2(x))$ where P_2 is a polynomial of degree at most two, and the bi-variate polynomial $y^2 + P_2(x)$ is irreducible. In this section we prove the following:

Theorem 1 *The reversible foliation $\mathcal{F}(df)$ is elliptic if and only if, after an affine change of the*

variables, it has a first integral of the form:

(rv1) $f = x^{-3}(y^2+ax^2+bx+c)$	(rv2) $f = x(y^2+cx^2+bx+a)$
(rv3) $f = x^{-3/2}(y^2+ax^2+bx+c)$	(rv4) $f = x^{-1/2}(y^2+cx^2+bx+a)$
(rv5) $f = x^{-4}(y^2+ax^2+bx+c)$	(rv6) $f = x^{2}(y^2+cx^2+bx+a)$
(rv7) $f = x^{-4/3}(y^2+bx+c)$	(rv8) $f = x^{-2/3}(y^2+cx^2+bx)$
(rv9) $f = x^{-4/3}(y^2+ax^2+bx)$	(rv10) $f = x^{-2/3}(y^2+bx+a)$
(rv11) $f = x^{-5/3}(y^2+ax^2+bx)$	(rv12) $f = x^{-1/3}(y^2+bx+a)$
(rv13) $f = x^{-5/4}(y^2+ax^2+bx)$	(rv14) $f = x^{-3/4}(y^2+bx+a)$
(rv15) $f = x^{-7/4}(y^2+ax^2+bx)$	(rv16) $f = x^{-1/4}(y^2+bx+a)$
(rv17) $f = x^{-5/2}(y^2+ax^2+bx)$	(rv18) $f = x^{1/2}(y^2+bx+a)$.

or

$$(i) f = x^{-1+\frac{2}{k}}(y^2+x), k \in \mathbb{Z}^* \setminus 2\mathbb{Z},$$

$$(ii) f = x^{-1+\frac{3}{k}}(y^2+x), k \in \mathbb{Z}^* \setminus 3\mathbb{Z}.$$

***Remark* 1** *We shall assume moreover that $c \neq 0$ for* (rv3), (rv4).

Proof. Let Γ_t be the set of $(x,y) \in \mathbb{C}^2$ such that for some determination of the multi-valued function x^λ holds $f(x,y) = t$. If the connected components of Γ_t for all t are algebraic curves, then $\lambda \in \mathbb{Q}$ and we put $\lambda = \frac{p}{q}$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}^*$ and $gcd(p,q) = 1$) with $P_2(x) = ax^2+bx+c \in \mathbb{C}_2[x]$.

As the foliation is reversible we may suppose that y^2+ax^2+bx+c is irreducible, or simply $b^2-4ac \neq 0$. We shall suppose first that $a \neq 0, c \neq 0$, that is to say the quadric $\{y^2+ax^2+bx+c=0\}$ is not tangent to the line at infinity in $\mathbb{P}^2$ and to the line $\{x=0\}$.

2.1.1 The case $a \neq 0, c \neq 0, b^2-4ac \neq 0$.

After a scaling of t and an affine transformation we may suppose that

$$f = x^\lambda(y^2+x^2+bx+c). \tag{6}$$

By abuse of notation we put

$$\Gamma_t = \{x^{p/q}(y^2+x^2+bx+c) = t\}$$

and in a similar way we define

$$\tilde{\Gamma}_t = \{X^p(Y^2+X^{2q}+bX^q+c) = t\}. \tag{7}$$

Lemma 1 *The map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2 : (X,Y) \to (x,y) = (X^q, Y)$ induces an isomorphism of Γ_t and $\tilde{\Gamma}_t$.*

Indeed, it is straightforward to check that $\varphi : \tilde{\Gamma}_t \to \Gamma_t$ is a bijection and therefore is a bi-holomorphic map.

To compute the genus of $\tilde{\Gamma}_t$ or Γ_t we distinguish two cases:

1. The case when $p < 0$.

   We obtain the hyper-elliptic curve $\{y^2 = -x^{2q} - bx^q + tx^{-p} - c\}$. The roots of the polynomial $-x^{2q} - bx^q + tx^{-p} - c$ are different and non zeros since t is generic. Consequently, its genus is one if and only if the degree of the polynomial is 3 or 4 and thus we get:

   (a) $f = x^{-3}(y^2+x^2+bx+c)$
   (b) $f = x^{-4}(y^2+x^2+bx+c)$
   (c) $f = x^{-\frac{1}{2}}(y^2+x^2+bx+c)$
   (d) $f = x^{-\frac{3}{2}}(y^2+x^2+bx+c)$

2. Suppose now $p \geq 0$.

   We easily have: $y^2 x^p = t - x^{2q+p} - bx^{q+p} - cx^p$. Thus after a birational transformation, we obtain:
$$y^2 = x^p(t - x^{2q+p} - bx^{q+p} - cx^p).$$
   Since t is generic, all the roots of $t - x^{2q+p} - bx^{q+p} - cx^p$ are different and do not vanish.

   - If p is even we have:
$$\left(\frac{y}{x^{\frac{p}{2}}}\right)^2 = t - x^{2q+p} - bx^{q+p} - cx^p.$$
     Consequently, it is elliptic when $2q + p$ either equal 3 or 4. This gives the following curves:

     (a) $x(y^2 + x^2 + bx + c) = t$
     (b) $x^2(y^2 + x^2 + bx + c) = t$.
   - If p is odd, then we have:
$$\left(\frac{y}{x^{\frac{p-1}{2}}}\right)^2 = x(t - x^{2q+p} - bx^{q+p} - cx^p).$$
     Since all the roots of $t - x^{2q+p} - bx^{q+p} - cx^p$ are different and non zeros, the curve is elliptic if and only $2q + p$ either equals 2 or 3, which gives the solution (b) above.

This we have obtained the cases (rv1)-(rv6) in Theorem 1.

2.1.2 The case $a \neq 0, c = 0, b^2 - 4ac \neq 0$.

This means that the quadric $\{y^2 + ax^2 + bx = 0\}$ is tangent to the line $\{x = 0\}$ and is transversal to the line at infinity, see Figure 7. After an affine transformation and scaling of t we get $P_2(x) = ax^2 + bx$ with $a \neq 0$. Therefore we need to compute the genus of $\{x^p(y^2 + x^{2q} + bx^q) = t\}$ for generic t.

If $p \geq 0$ the same computations as case $a \neq 0, c \neq 0$ give the same solutions of the problem.
Let $p < 0$ and suppose that $p = 2a$ is even. We have $(x^a y)^2 = -x^{2q+p} - bx^{q+p} + t$, so if $-p \leq q$, it has genus one if and only $2q + p = 3$ or 4, hence $q \leq 4$ and $(p, q) = (-2, 3)$.
If $-p \geq 2q$ the curve above is birational to the curve $y^2 = -x^{-p-q} - bx^{-p-2q} + t$ and so it has genus one if $-p - q = 3$ or 4 which leads to $(p, q) = (-4, 1)$.
If $q < -p < 2q$, because p is even and q is odd, it is equivalent to calculate the genus of $\{y^2 = x(tx^{-(q+p)} - x^q - b)\}$. Therefore $(p, q) = (-4, 3)$.

Now suppose that $p = 2a + 1$ is odd. The curve is birationally equivalent to $\{y^2 = -x(x^{2q+p} + bx^{q+p} - t)\}$. As above we get: $(p, q) = (-1, 2)$, $(-3, 1)$,$(-5, 2)$, $(-5, 3)$, $(-5, 4)$, $(-7, 4)$.

To resume, we proved

Proposition 1 *The foliation $\mathcal{F}(x^\lambda(y^2 + x^2 + bx))$ is elliptic if and only it has under affine transformation a first integral of the kind :*

$$f = x(y^2 + x^2 + bx) \qquad f = x^2(y^2 + x^2 + bx)$$

$$f = x^{-3}(y^2 + x^2 + bx) \qquad f = x^{-4}(y^2 + x^2 + bx)$$

$$f = x^{-\frac{1}{2}}(y^2 + x^2 + bx) \qquad f = x^{-\frac{5}{2}}(y^2 + x^2 + bx)$$

$$f = x^{-\frac{2}{3}}(y^2 + x^2 + bx) \qquad f = x^{-\frac{4}{3}}(y^2 + x^2 + bx)$$

$$f = x^{-\frac{5}{4}}(y^2 + x^2 + bx) \qquad f = x^{-\frac{7}{4}}(y^2 + x^2 + bx)$$

$$f = x^{-\frac{5}{3}}(y^2 + x^2 + bx)$$

Here we get (rv1) – (rv6) except (rv3) according to Remark 1, (rv8) and the end of the left column of Theorem 1.

2.1.3 $a = 0, c \neq 0,\ b^2 - 4ac \neq 0.$

This means that the quadric $\{y^2 + bx + c = 0\}$ is tangent to the line at infinity and is transversal to the line $\{x = 0\}$. The birational change of variables $x \to 1/x$, $y \to y/x$ shows that this is equivalent to the case $a \neq 0, c = 0$ and we get:

Proposition 2 *The foliation $\mathcal{F}(x^\lambda(y^2 + x + c))$ is elliptic if and only it has under affine transformation a first integral of the kind :*

$$f = x(y^2 + x + c) \qquad f = x^2(y^2 + x + c)$$

$$f = x^{-3}(y^2 + x + c) \qquad f = x^{-4}(y^2 + x + c)$$

$$f = x^{-\frac{3}{2}}(y^2 + x + c) \qquad f = x^{\frac{1}{2}}(y^2 + x + c)$$

$$f = x^{-\frac{2}{3}}(y^2 + x + c) \qquad f = x^{-\frac{4}{3}}(y^2 + x + c)$$

$$f = x^{-\frac{1}{4}}(y^2 + x + c) \qquad f = x^{-\frac{3}{4}}(y^2 + x + c)$$

$$f = x^{-\frac{1}{3}}(y^2 + x + c)$$

Here we get (rv1) – (rv6) except (rv4) according to Remark 1, (rv7) and the end of the right column of Theorem 1.

2.1.4 $a = c = 0, b \neq 0$

This means that the quadric $\{y^2 + ax^2 + bx + c = 0\}$ is tangent to the line at infinity and to the line $\{x = 0\}$. Up to affine change of re-scalings we may suppose $f = x^{p/q}(y^2 + x)$.

If p is even the curve $\tilde{\Gamma}_t$ is birational to $y^2 = -x^{q+p} + t$.
If p is odd the curve $\tilde{\Gamma}_t$ is birational to $y^2 = -x(x^{q+p} - t)$.
For $p \geq 0$ this curve is elliptic if and only: $(p, q) = (1, 1)$, $(2, 1)$ and $(1, 2)$.
Now, if $p \leq 0$ and $q + p \geq 0$ the conditions are $q + p = 2$, 3 or 4 with p even.
The case $q + p \leq 0$ gives similarly$-q - p = 2$, 3 or 4 with p even.
Notice that we must have q prime with the integers 2 or 3 or 4 when considering all cases. This gives the following:

Proposition 3 *The foliation $\mathcal{F}(x^\lambda(y^2 + bx))$ with $b \neq 0$ is elliptic if and only if it has a first integral of the kind :*

$$f = x^{-1+\frac{2}{k}}(y^2 + x),\ k \in \mathbb{Z}^* \setminus 2\mathbb{Z} \qquad f = x^{-1+\frac{3}{k}}(y^2 + x),\ k \in \mathbb{Z}^* \setminus 3\mathbb{Z}$$

Finally, Theorem 1 is proved.

$\square$

2.2 The Lotka-Volterra case

Theorem 2 *The Lotka-Volterra foliation $\mathcal{F}(df)$ is elliptic if and only if, after an affine change of the variables, it has a first integral of the form:*

$$\begin{array}{lll} \text{(lv1) } f = x^2y^3(1-x-y) & \text{(lv2) } f = x^{-6}y^2(1-x-y) & \text{(lv3) } f = x^{-6}y^3(1-x-y) \\ & \text{(lv4) } f = x^{-4}y^2(1-x-y) & \\ & \text{(lv5) } f = x^{-6}y^3(1-x-y)^2 & \end{array}$$

or

$$(iii) f = x^{\frac{3}{k}} y^{\frac{l}{k}}(1+y), \qquad (iv) f = x^{-1+\frac{3}{k}-\frac{l}{k}} y^{\frac{l}{k}}(x+y)$$
with $k \in \mathbb{Z}^* \setminus 3\mathbb{Z}$ *and* $l - k \in 3\mathbb{Z}$,
$$(v) f = x^{\frac{4}{k}} y^{\frac{l}{k}}(1+y), \qquad (vi) f = x^{-1+\frac{4}{k}-\frac{l}{k}} y^{\frac{l}{k}}(x+y)$$
with $k \in \mathbb{Z}^* \setminus 2\mathbb{Z}$ *and* $l - k \in 4\mathbb{Z}$
$$(vii) f = x^{\frac{6}{k}} y^{\frac{l}{k}}(1+y), \qquad (viii) f = x^{-1+\frac{6}{k}-\frac{l}{k}} y^{\frac{l}{k}}(x+y)$$
with $k \in \mathbb{Z}^*, l \in 2\mathbb{Z}$ *and* $kl - 2 \in 6\mathbb{Z}$
$$(ix) f = x^{\frac{4}{k}} y^{\frac{2l}{k}}(1+y) \qquad (x) f = x^{-1+\frac{4}{k}-\frac{2l}{k}} y^{\frac{2l}{k}}(1+y)$$
with $k, l \in \mathbb{Z}^* \setminus 2\mathbb{Z}$
$$(xi) f = x^{\frac{6}{k}} y^{\frac{3l}{k}}(1+y) \qquad (xii) f = x^{-1+\frac{6}{k}-\frac{3l}{k}} y^{\frac{3l}{k}}(1+y)$$
with $k \in \mathbb{Z}^* \setminus 3\mathbb{Z}$ *and* $l \in \mathbb{Z}^* \setminus 2\mathbb{Z}$

and moreover $\mathrm{gcd(k,l)} = 1$.

Proof. An algebraic first integral is given by

$$f = x^{p_1} y^{p_2}(ax+by+c)^r, p_1, p_2 \in \mathbb{Z}, r \in \mathbb{N}^*.$$

This defines a divisor in $\mathbb{P}^2$: $D = p_1 L_1 + p_2 L_2 + r L_3$ where $L_i, i = 1..3$ are projective lines. As in previous section, the study below will depend on the geometry of the reduced divisor $\tilde{D}$ (i.e without multiplicities) associated to D. First we will consider the generic case, that is the projective lines $L_i, i = 1..3$ have normal crossings toward each other.

2.2.1 The case $a \neq 0,\ b \neq 0,\ c \neq 0$.

First we may suppose under affine transformation $b = c = -a = 1$.
The expression of $\tilde{D}$ invites us to divide the study in 4:

$$p_1 > 0, p_2 > 0 \tag{8}$$

$$p_1 < 0, p_2 > 0, p_1 + p_2 + q > 0 \tag{9}$$

$$p_1 < 0, p_2 > 0, p_1 + p_2 + q < 0 \tag{10}$$

$$p_1 < 0, p_2 > 0, p_1 + p_2 + q = 0. \tag{11}$$

In the shape of (10) the generic leaf is birational to the algebraic curve $X^{p_1} Y^{p_2} = t$ which is rational. Hence the generic case will be an obvious consequence of the 3 following propositions:

Proposition 4 . *The algebraic curve* $x^p y^q (1-x-y)^r = 1$ *with* $0 \leq p \leq q \leq r$, $\gcd(p,q,r) = 1$ *is of genus one if and only if* $(p,q,r) = (1,1,1)$ *or* $(1,1,2)$ *or* $(1,2,3)$.

Proposition 5 *The algebraic curve* $y^q(1-x-y)^r = x^p$ *with* $p,q,r > 0$, $-p+q+r < 0$ $\gcd(p,q,r) = 1$ *is of genus one if and only if* $(p,q,r) = (1,2,2)$ *or* $(3,2,2)$.

Proposition 6 *The algebraic curve* $y^q(1-x-y)^r = x^p$ *with* $p > 0, q > r > 0$, $-p+q+r > 0$ $\gcd(p,q,r) = 1$ *is elliptic if and only if* $(p,q,r) = (3,1,1), (4,1,1), (4,2,1), (6,2,1), (6,3,1)$ *or* $(6,3,2)$.

Proof of Proposition 4.
Let ω be a one form on a compact Riemann surface S. We write $\omega = \sum_i a_i P_i$ with P_i points of S. This sum is finite and we define the *degree* of ω : $\deg(\omega) = \sum_\mathrm{i} \mathrm{a_i}$. According to the *Poincaré-Hopf* formula (see [GH78]), any 1-form ω on S satisfies:

$$\deg(\omega) = 2g - 2. \tag{12}$$

Now, we use this formula with the riemann surface $\tilde{C}$ obtained after desingularisation of the irreducible algebraic curve C defined by the equation $x^p y^q (1-x-y)^r = 1$. Let $\pi : \tilde{C} \to C$ be such a desingularisation map. We compute below the degree of the one-form $\pi^*\omega$ where (by abuse of notation):

$$\omega = -\frac{dx}{x[q - qx - (q+r)y]} = \frac{dy}{y[p - py - (p+r)x]}.$$

The 1-form above has been chosen such that it has nor zeros nor poles outside the singular locus of C. Yet, C is only singular in the three singular points meeting the line at infinity: $[1:0:0]$, $[0:1:0]$, $[1:-1:0]$.
First we investigate the local behavior of ω near $[0:1:0]$. We get local coordinates near $[1:0:0]$ as follows:
Write $x = \frac{1}{u}$ with $u \to 0$.
After this change of coordinates, the equation becomes:

$$y^q (u - 1 - yu)^r = u^{p+r}.$$

Since $u \to 0$, we have the $m = gcd(q, p+r)$ different parametrisations of the m local branches near this point:

$$u = t^{\frac{q}{m}}$$
$$y = -e^{\frac{2ik\pi}{m}} t^{\frac{p+r}{m}} (1 + o(t^{\frac{p+r}{m}})). \quad k = 0 \ldots m-1.$$

For each branch, locally,

$$\omega = -\frac{q}{m} t^{\frac{q}{m}-1} (1 + o(t^{\frac{q}{m}-1})) dt.$$

Finally, for $\pi^*\omega$ we get after a finite number of blowing-ups m points where our 1-form has a zero of order $\frac{q}{m} - 1$.

The study is completely similar for the remaining singular points: near $[0:1:0]$ we obtain $n = gcd(p, q+r)$ points where the 1-form $\pi^*\omega$ has a zero of order $\frac{p}{n} - 1$ and near $[1:-1:0]$, we have $l = gcd(r, p+q)$ points where $\pi^*\omega$ has a zero of order $\frac{r}{l} - 1$.

Finally, the numbers involved satisfy the following relation:

$$p + q + r - m - n - l = 2g - 2 \tag{13}$$

and consequently, this curve is elliptic when:

$$p + q + r = m + n + l. \tag{14}$$

Now we have to resolve this diophantine equation:

We always have: $m \leq q$, $n \leq p$ and $l \leq r$. Hence (14) is true if and only if:

$$gcd(q, r+p) = q;$$
$$gcd(p, q+r) = p;$$
$$gcd(r, p+q) = r.$$

Let $\alpha,\ \beta,\ \gamma \in \mathbb{N}^*$ such that:

$$r + p = q\alpha; \tag{a}$$
$$r + q = p\beta; \tag{b}$$
$$p + q = r\gamma. \tag{c}$$

Using (a) and (b), we obtain $(\alpha+1)q = (\beta+1)p$.
Using (b) and (c), we obtain $(\gamma\beta - 1)q = (\gamma+1)p$.

Hence we have:

$$\frac{q}{p} = \frac{\beta+1}{\alpha+1} = \frac{\gamma\beta-1}{\gamma+1}$$

which gives the following equation:

$$\alpha\beta\gamma = 2 + \alpha + \beta + \gamma. \tag{15}$$

The solutions of this equation are under symmetry $(2,2,2),(3,3,1)$ and $(5,2,1)$ which gives at last the solutions (p,q,r) of Proposition 4.

□

Proof of Proposition 5.
The proof is similar. We still use (12) with ω a $1-forme$ that without both zeros and poles outside the singular locus of the algebraic curve C defined by $y^q(1-x-y)^r = x^p$:

$$\omega = -\frac{dx}{x[q-qx-(q+r)y]} = \frac{dy}{y[-p+py-(r-p)x]}.$$

Here the singular locus is no longer as before. It has two singular points at infinity: $[1:0:0]$ et $[1:-1:0]$ and moreover $(0,0)$ et $(0,1)$ in the affine chart. Local considerations as above naturally leads us to the following

Lemma 2 *The irreducible algebraic curve above has genus one if and only p, q and r satisfy the equation:*

$$q + r = gcd(p,q) + gcd(q,r) + gcd(r, r+q-p) + gcd(q, r+q-p). \tag{16}$$

Writing $m = gcd(p,q)$, $n = gcd(q,r)$, $l = gcd(r, r+q-p)$ and $s = gcd(q, r+q-p)$, then there exists integers $\alpha,\beta,\gamma,\delta$ such that $r = n\alpha = l\beta$ and $q = s\gamma = m\delta$ so that (16) is equivalent to:

$$q + r = \frac{\gamma+\delta}{\gamma\delta}q + \frac{\alpha+\beta}{\alpha\beta}r \tag{17}$$

Hence we have $\gamma+\delta = \gamma\delta$ and $\alpha+\beta = \alpha\beta$ and therefore $\alpha = \beta = \gamma = \delta = 2$.Hence $m = s$ and as m divides p and q then m divides r and finally $m = 1$. Similarly, we get $n = 1$ and consequently $q = r = 2$. Now, remind that $p < r + q = 4$ so that p either equals 1 or 3 (2 is excluded as $\gcd(p,q,r) = 1$). Now we easily verify that $(1,2,2)$ and $(3,2,2)$ are the solutions to the equation 16) above which proves Proposition 5.

□

Proof of Proposition 6.
After the birational change of variable: $x \to \frac{1}{x}$ and $y \to \frac{y}{x}$, the genus (which is a birationnal invariant for curves) is the same as the genus of the algebraic curve:

$$x^{p-r-q}y^q(1-x-y)^r = 1.$$

Then this is an immediate consequence of Proposition 4 above.

□

Consequently we found $(\mathrm{lv}1-5)$ of Theorem 2.

2.2.2 The case $a = 0$.

Under affine transformation we may suppose $b = c = 1$. Geometrically $\{y = 0\}$ and $\{y = 1\}$ both intersect at infinity. Notice first the following:

$$x^{\lambda} y^{\mu} (1+y) = (xy^n)^{\lambda} y^{\mu - n\lambda} (1+y)$$

for $n \in \mathbb{Z}$, hence $x^{\lambda} y^{\mu}(1+y) = t$ is birational to $x^{\lambda} y^{\mu - n\lambda}(1+y) = t$ so that we only need to study when λ and μ are strictly positives. This naturally leads to the following:

Proposition 7 *The algebraic curve:*

$$C = \{(x,y) \in \mathbb{C}^2, x^p y^q (1+y)^r = 1\}$$

with $0 \le r \le q$ and $0 \le p$, where $gcd(p,q,r) = 1$. is elliptic if and only, under permutations of $\{y = 0\}$ and $\{y+1 = 0\}$ it is in the following list:

$$\begin{array}{lll} x^3 y^{1+3u}(1+y)^{1+3v} = 1; & x^3 y^{2+3u}(1+y)^{2+3v} = 1 & \\ x^4 y^{1+4u}(1+y)^{1+4v} = 1; & x^4 y^{3+4u}(1+y)^{3+4v} = 1; & x^4 y^{2(1+2u)}(1+y)^r = 1, r \in Z^* \setminus 2\mathbb{Z} \\ x^6 y^{2+6u}(1+y)^{1+6v} = 1; & x^6 y^{5+6u}(1+y)^{4+6v} = 1; & x^6 y^{3(2u+1)}(1+y)^r = 1, r \in \mathbb{Z}^* \setminus 3\mathbb{Z}. \end{array}$$

Proof of Proposition 7.
We still use (12) with a judiciously chosen ω without zeros nor poles in its regular locus:

$$\omega = \frac{dx}{x(q + (q+r))y} = -\frac{dy}{py(1+y)}.$$

This curve has two points at infinity, namely $[1:0:0]$ and $[0:1:0]$, where C is singular (C is regular in the affine chart)
Near $[1:0:0]$, we have two branches where a local equation of each is respectively:

$$Y^q = u^p$$

$$(1+Y)^r = u^p$$

where $x = \frac{1}{u}$. Thus, writing: $m = pgcd(p,q)$ and $n = pgcd(p,r)$, we obtain the parametrisations:

$$Y = t^{\frac{p}{m}}$$

$$u = t^{\frac{q}{m}}$$

and

$$Y = t^{\frac{p}{n}}$$

$$u = t^{\frac{r}{n}}.$$

Both give a pole of order 1 for $\pi^* \omega$, hence we obtain, adding up the different possible parametrisations, $-m-n$ in the Poincaré-Hopf formula.
A similar calculus near the other point at infinity gives a zero of order $\frac{p}{l} - 1$ where $l = (p, q+r)$ with l different parametrisations.
Thus we finally obtain the equality:

$$p = m + n + l. \tag{18}$$

We want to resolve this equation. Consider:

$$p = n\gamma$$

$$p = m\beta$$

$$p = l\alpha$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}^*$. Then (18) is equivalent to the following well-known equation:

$$\alpha\beta\gamma = \alpha\beta + \alpha\gamma + \beta\gamma. \tag{19}$$

The solutions are up to permutation:

$$(3,3,3)$$
$$(2,4,4)$$
$$(2,3,6).$$

The solution $(3,3,3)$ implies $m = n = l$.
As $\gcd(p,q,r) = 1$ we have: $m = n = l = 1$ and so $p = 3$, $1 = (r,3)$, $1 = (q,3)$. Hence $r = 1\ ,\ 2 \mod (3)$ and so does q. Finally, reminding $l = \gcd(p, q+r)$, we conclude that $(p,q,r) = (3, 1+3u, 1+3v), (3, 2+3u, 2+3v)$.

The solution $(2,4,4)$ implies $l = 2n$ and $m = n = 1$ and the same argument shows that $(p,q,r) = (4, 1+4u, 1+4v)$ or $(4, 3+4u, 3+4v)$. There are 2 other solutions (permutations of $(2,4,4)$). Under permutations of the two lines $\{y = 0\}$ and $\{y+1 = 0\}$, we only need to study $(4,2,4)$. A similar resolution thus gives $(p,q,r) = (4, 2(1+2u), 1+2v)$.

The solution $(2,3,6)$ implies $m = 2$, $n = 1$, $l = 3$, so $(p,q,r) = (6, 2+6u, 1+6v)$ or $(6, 5+6u, 4+6v)$. As above, we need to take under consideration the solutions $(3,2,6)$ and $(6,2,3)$ wich respectively gives $(p,q,r) = (6, 3(2u+1), r)$ with $\gcd(r,3) = 1$ and $(6, 2(3u+1), 6v+1)$ or $(6, 2(3u+2), 6v+5)$. Finally the proposition is proved.

□

This we obtain the last cases of the left column of Theorem 2.

2.2.3 The case $c = 0$.

Under affine transformation we may suppose $a = b = 1$.
Here the three lines $\{x = 0\}$, $\{y = 0\}$ and $\{x + y = 0\}$ intersect themselves at the origine.
Now, the algebraic curve $x^\lambda y^\mu (x+y) = t$ is obviously birational to $x^{\lambda+\mu+1} y^\mu (1+y) = t$, so this case falls from the preceding results and we get the last cases of the right column of Theorem 2.

We have investigated in fact all the possible first integrals. Indeed, if our foliation admits a first integral: $f = x^{-\alpha} y^{-\beta} (ax + by + c)$ with α, β real positive numbers, then after affine transformation it has a first integral: $g = X^{-\frac{1}{\alpha}} Y^{\frac{\beta}{\alpha}} (AX + BY + C)$. Hence Theorem 2 is proved.

□

2.3 The reversible Lotka-Voltera case

Theorem 3 *The reversible Lotka-Voltera foliation $\mathcal{F}(df)$ is elliptic if and only if, after an affine change of the variables, it has a first integral of the form*

$(\mathrm{rlv1})\ \mathrm{f} = \mathrm{xy}(1 - \mathrm{x} - \mathrm{y})$ $\quad(\mathrm{rlv2})\ \mathrm{f} = \mathrm{x}^{-3}\mathrm{y}(1 - \mathrm{x} - \mathrm{y})$
$(\mathrm{rlv3})\ \mathrm{f} = \mathrm{x}^{2}\mathrm{y}(1 - \mathrm{x} - \mathrm{y})$ $\quad(\mathrm{rlv4})\ \mathrm{f} = \mathrm{x}^{-4}\mathrm{y}(1 - \mathrm{x} - \mathrm{y})$
$(\mathrm{rlv5})\ \mathrm{f} = \mathrm{x}^{-3}\mathrm{y}^{2}(1 - \mathrm{x} - \mathrm{y})^{2}$ $\quad(\mathrm{rlv6})\ \mathrm{f} = \mathrm{x}^{-1}\mathrm{y}^{2}(1 - \mathrm{x} - \mathrm{y})^{2}$

or :
$(xiii) f = x^{\frac{3}{k}}(y+1)(y-1), k \in \mathbb{Z}^* \setminus 3\mathbb{Z}$ $\quad(x1v) f = x^{-2+\frac{3}{k}}(x+y)(x-y), k \in \mathbb{Z}^* \setminus 3\mathbb{Z}$
$(xv) f = x^{\frac{4}{k}}(y+1)(y-1), k \in \mathbb{Z}^* \setminus 2\mathbb{Z}$ $\quad(xvi) f = x^{-2+\frac{4}{k}}(x+y)(x-y), k \in \mathbb{Z}^* \setminus 2\mathbb{Z}$

Proof.
Without loss of generality we may suppose that the foliation has a first integral of the form $f = x^\lambda (y^2 + a(x - \frac{b}{2a})^2)$ if $a \neq 0$ and $f = x^\lambda (y^2 + c)$ otherwise (we deal about quadratic foliations so that c necessary does not vanish). For $a = 0$ this a consequence of Proposition 7. Now look at $a \neq 0$:

if $b = 0$, the curve $x^\lambda(y^2 + ax^2) = t$ is birational to $x^{\lambda+2}(y^2 + 1) = t$ thus the conditions are $p + 2q = 3, 4$ or $-2q - p = 3, 4$.
if $b \neq 0$ the quadric is a reducible polynomial so that this case is a straightforward consequence of Propositions 4, 5 and 6. We notice that the last cases of reversible Lotka-Volterra are exactly the last cases of Lotka-Volterra under the condition $l = k$ when it is possible (For (vii) and $(viii)$ of Theorem 2 we can't have $k = l$). This gives Theorem 3.

□

Notice that the case $b = 0$ is also a consequence of the degenerate Lotka-Voltera case with the three invariant lines involved intersecting themselves, but the calculus is here so easy that we proved it directly and is useful to test our preceeding calculus.

3 Topology of the singular fibers and Kodaira's classification

Now we focus on the singular fibers of the induced elliptic surfaces. First of all, Recall that two birational elliptic surfaces have the same minimal model (see [Ka75, M89]). Some of our previous elliptic surfaces are obviously birationnals and therefore have the same singular fibers under permutation. First we investigate such mappings. Then we give some examples of computation of the singular fibers to illustrate the way we obtained Tables 2, 3, 4, 5, 6, 7, 8.

3.1 The reversible case

3.1.1 Birational mappings

The first integrals are given by the algebraic equation:

$$x^\lambda(y^2 + ax^2 + bx + c) = t$$

with a, b, c complex numbers satisfying some conditions and λ a rational number. We have an easy birational mapping (we already used it, see Section 2.1.3): $X = \frac{1}{x}$, $Y = \frac{y}{x}$ which leads to

$$x^{-2-\lambda}(y^2 + cx^2 + bx + a) = t.$$

When considering this mapping in $\mathbb{P}^2$ with homogeneous coordinates $[x : y : z]$ this last permutes in fact the projective lines $\{x = 0\}$ and $\{z = 0\}$. Thus for each line of Table 1 we only need to study either the right or the left element.

For degenerate cases, notice the change of variables $(X, Y) = (xy, y)$ birationnally leads (i) (resp. (ii)) to (x) (resp. (xii)) Lotka-Volterra elliptic case with $l = 1$. Consequently, such cases will be a consequence of the calculus of the singular fibers of the Lotka-Volterra cases (see below).

Be careful that the geometry of the divisors appearing in the first integrals (including the line at infinity) is of importance as we shall blow-up the indetermination points. Birationally, the different geometrical description of the divisors in the reversible case are the following:

(1) The divisors are in general position (see Figure 1). This concerns (rv2), (rv4), (rv6) with $a, b, c \neq 0$.

(2) $\{Q = 0\}$ and $\{x = 0\}$ are in general position and $\{Q = 0\}$ and $\{z = 0\}$ have only one tangent double point (see Figure 7). This concerns (rv2) with $c = 0$, (rv3) with $a = 0$, (rv6) with $c = 0$, (rv7), (rv10), (rv12), (rv14), (rv16).

(3) Both projective lines $\{x = 0\}$ and $\{z = 0\}$ have a double tangent point with the quadric . This concerns (i) and (ii).

$\lambda < -1$	$\lambda > -1$
$(rv1)$ $x^{-3}(y^2+ax^2+bx+c)=t$	$(rv2)$ $x(y^2+cx^2+bx+a)=t$
$(rv3)$ $x^{-\frac{3}{2}}(y^2+ax^2+bx+c)=t, c\neq 0$	$(rv4)$ $x^{-\frac{1}{2}}(y^2+cx^2+bx+a)=t, c\neq 0$
$(rv5)$ $x^{-4}(y^2+ax^2+bx+c)=t$	$(rv6)$ $x^2(y^2+cx^2+bx+a)=t$
$(rv7)$ $x^{-\frac{4}{3}}(y^2+bx+c)=t$	$(rv8)$ $x^{-\frac{2}{3}}(y^2+cx^2+bx)=t$
$(rv9)$ $x^{-\frac{4}{3}}(y^2+ax^2+bx)=t$	$(rv10)$ $x^{-\frac{2}{3}}(y^2+bx+a)=t$
$(rv11)$ $x^{-\frac{5}{3}}(y^2+ax^2+bx)=t$	$(rv12)$ $x^{-\frac{1}{3}}(y^2+bx+a)=t$
$(rv13)$ $x^{-\frac{5}{4}}(y^2+ax^2+bx)=t$	$(rv14)$ $x^{-\frac{3}{4}}(y^2+bx+a)=t$
$(rv15)$ $x^{-\frac{7}{4}}(y^2+ax^2+bx)=t$	$(rv16)$ $x^{-\frac{1}{4}}(y^2+bx+a)=t$
$(rv17)$ $x^{-\frac{5}{2}}(y^2+ax^2+bx)=t$	$(rv18)$ $x^{\frac{1}{2}}(y^2+bx+a)=t$

(i) $x^{-1+\frac{2}{k}}(y^2+x)=t$	$k\in\mathbb{Z}^*\setminus 2\mathbb{Z}$
(ii) $x^{-1+\frac{3}{k}}(y^2+x)=t$	$k\in\mathbb{Z}^*\setminus 3\mathbb{Z}$

Table 1: The elliptic reversible cases.

3.1.2 The singular fibers $t = 0$ and $t = \infty$

Here we illustrate the results with examples:

EXAMPLE 1: The singular fibers of (rv4)*:*

Embedding our first integral in $\mathbb{P}^2$, one have:

$$\frac{(y^2 + ax^2 + bxz + cz^2)^2}{xz^3} = t.$$

Geometrically, there are two lines: $\{x = 0\}$ and $\{z = 0\}$ with multiplicities respectively -1 and -3 that intersect in $[0 : 1 : 0]$, and a conic that intersects both lines in four points, namely $A_1 = [0 : \sqrt{-c} : 1]$, $A_2 = [0 : -\sqrt{-c} : 1]$, $B_1 = [\sqrt{-a} : 1 : 0]$, $B_2 = [-\sqrt{-a} : 1 : 0]$ with normal crossing each time (see Figure 1).

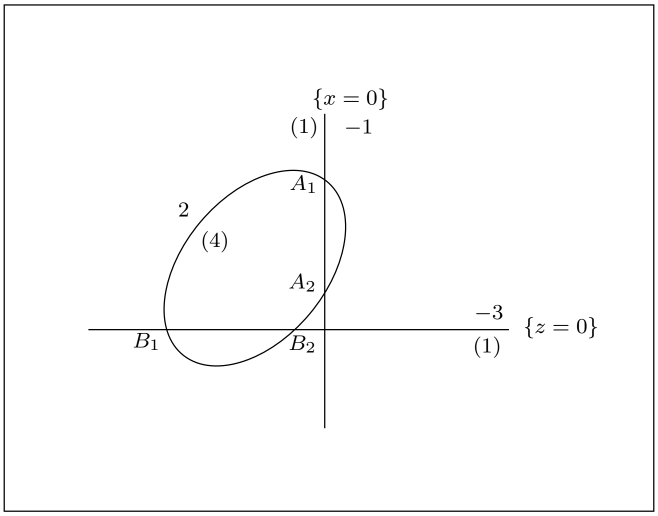

Figure 1: Geometrical situation of (rv4).

The rational function is not defined in these four points, thus we need to blow-up them.

- Near A_1
  In local coordinates the rational function becomes: $\frac{Y^2}{X}$.
  We need two blowing-ups to define the rational function near this point:

  $$\frac{Y^2}{X} \to \frac{Y}{X} \to \text{separation of both local branches}$$

  Remind that blowing a point of $\mathbb{P}^2$ that belongs to a divisor D decreases the self-intersection of D by one (see [GH78] for example). Writing the self-intersection and the multiplicities (the self-intersection numbers are inside the ()) we get the situation of Figure 2.

  We study A_2 along the same lines. See Figure 3.

- Near B_1.
  Locally the rational function becomes $\frac{Y^2}{Z^3}$.
  The successive blowing-ups give the following local equations until separation:

  $$\frac{Y^2}{Z^3} \to \frac{Y^2}{Z} \to \frac{Y}{Z} \to \text{separation of the branches}$$

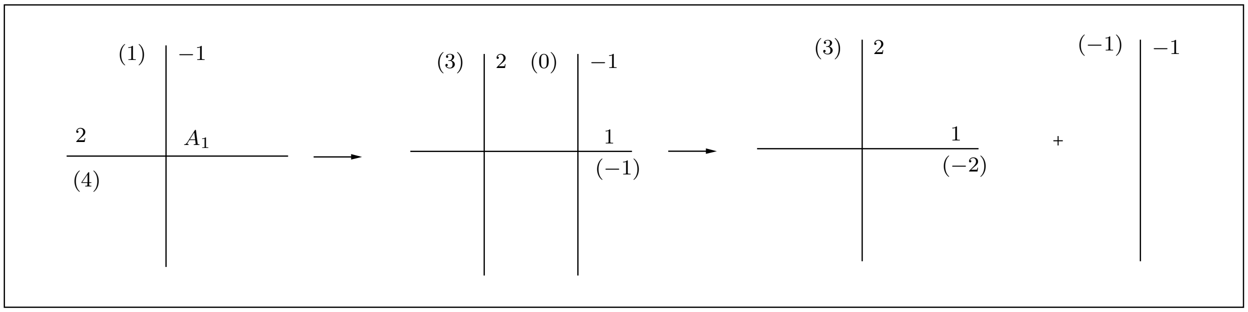

Figure 2: The successive blowing-ups of A_1.

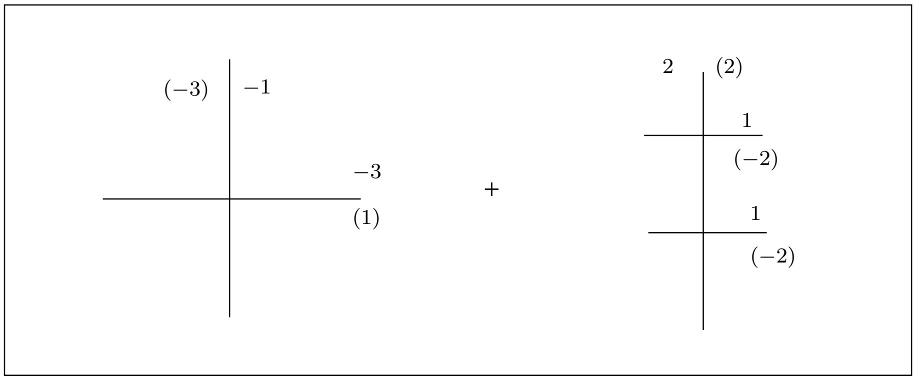

Figure 3: Summary ot the situation after blowing-up A_1 and A_2.

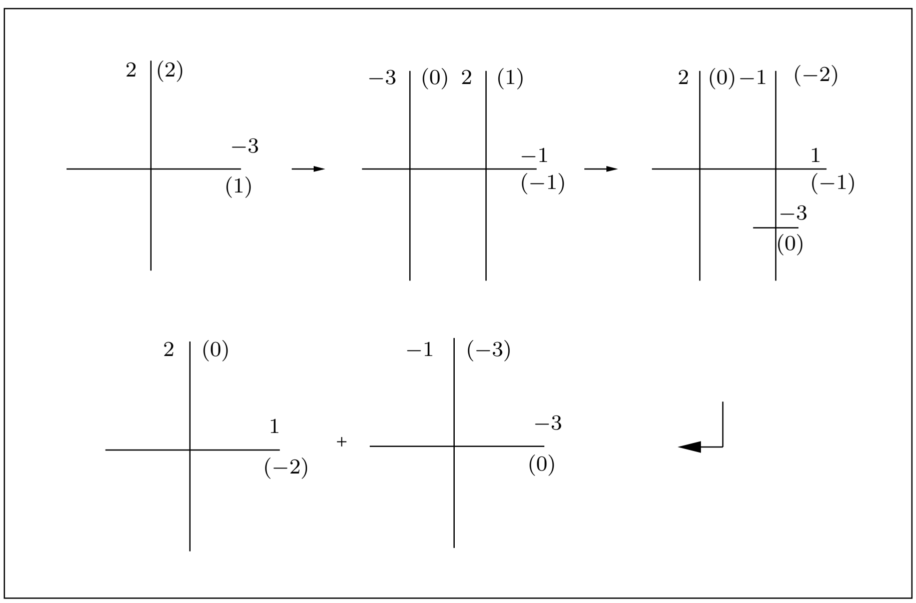

Figure 4: The successive blowing-ups of B_1.

see Figure 4.

The situation is still the same near B_2. We obtain 2 singular fibers. See Figure 5.

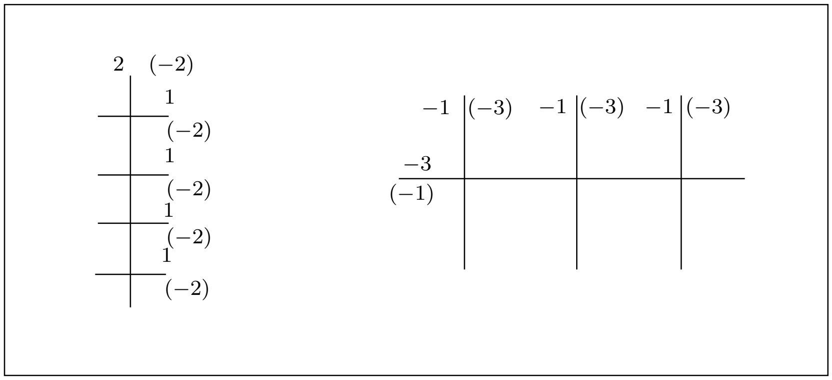

Figure 5: The fibers $t = 0$ and $t = \infty$ of (rv4).

We now have to recognize these singular fibers in Kodaira's classification (see [Ko63]). The singular fiber $t = 0$ is I_0^*, but we don't recognize the other. This is because we have branches with self-intersection -1. Recall that Kodaira's classification involves *minimal* elliptic surfaces i.e no fiber contains an exceptionnal curve of the first kind.
Finally we get IV (the singular fiber at infinity). See Figure 6.

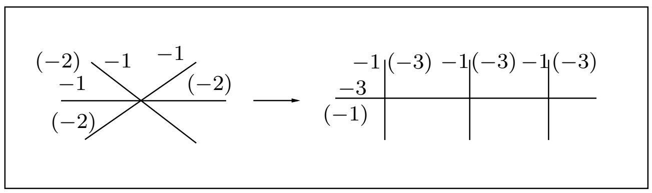

Figure 6: Contraction of the fiber $t = \infty$ of (rv4).

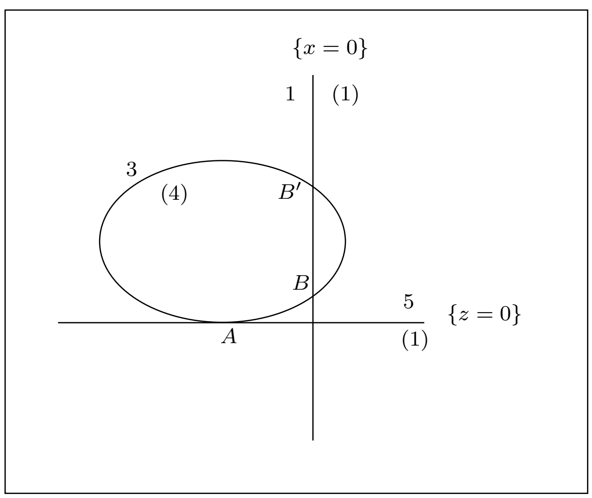

Figure 7: The divisor associated to (rv12).

EXAMPLE 2 : The singular fibers of (rv12) *:*

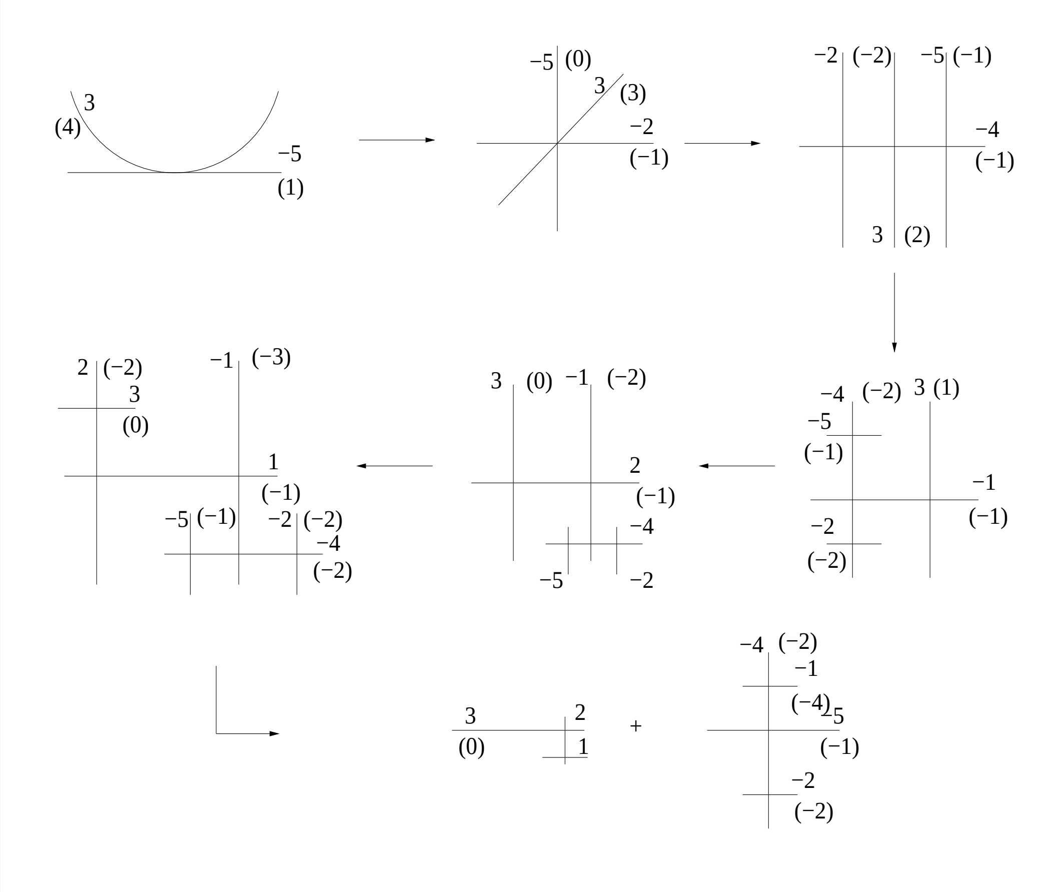

Figure 8: Successive blowing-ups of (rv12) near A.

The rational funcion here is:

$$\frac{(y^2+bxz+az^2)^3}{xz^5}.$$

The geometrical situation is explained in Figure 7.

- Study near A:
  Locally the rational function becomes: $\frac{Y^3}{Z^5}$. To begin with, we have:

$$\frac{(Y^2+Z)^3}{Z^5} \to \frac{(Y+Z)^3}{Z^5Y^2} \to \frac{(Z+1)^3}{Z^5Y^4} \to \text{separation of both local branches}$$

  Next we have to blow-up the point with local coordinate: $(Y=0, Z=-1)$, what gives locally :

$$\frac{Z^3}{Y^4} \to \frac{Z^3}{Y} \to \frac{Z^2}{Y} \to \frac{Z}{Y} \to \text{separation of both local branches.}$$

  the geometrical explanations are given in Figure 8.

- Study near B:
  Locally the rational function becomes: $\frac{Y^3}{X}$. Such a calculus has already been done in Example 1.

We finally obtain two fibers (see Figure 9).
For $t=0$ we recognize $IV*$. For $t=\infty$, one have to contract divisors with self-intersection -1 as in Figure 10. We finally get III of Kodaira's classification.

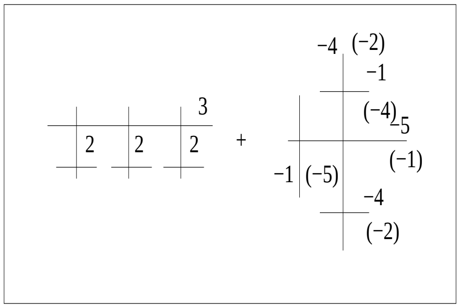

Figure 9: The fibers at $t = 0$ and $t = \infty$ of (rv12).

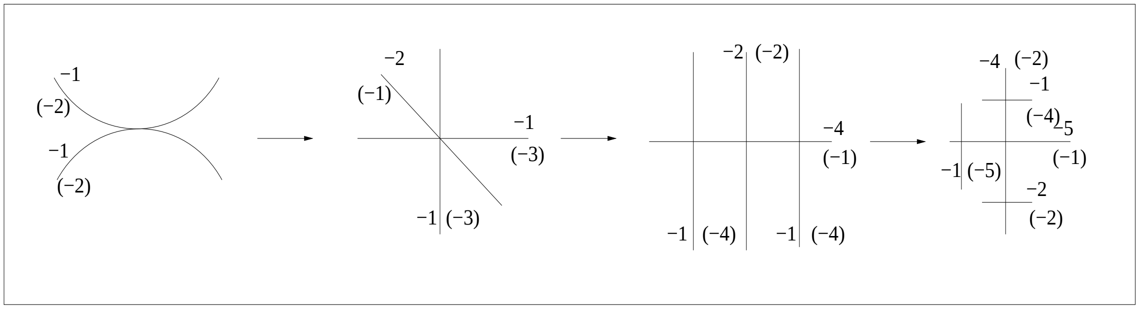

Figure 10: Contraction of divisors with self-intersection -1 for the fiber at infinity of (rv12).

3.1.3 Other(s) singular fiber(s)

Be careful that we do not have the full list of singular fibers: we also have to consider the singular points of our foliation whiches do not intersect both lines and the conic curve above. Here, the results concern the whole class of reversible systems inducing elliptic fibrations:
Writing:

$$f = x^{\lambda}(y^2 + ax^2 + bx + c),$$

(x, y) is a singular point if and only :

$$x^{\lambda} y = 0$$

$$x^{\lambda-1}(\lambda y^2 + (\lambda + 2)ax^2 + (\lambda + 1)bx + \lambda c) = 0.$$

Consequently (x_0, y_0) is a singular point which does not intersect both lines and the conic curve above if and only:

$$y_0 = 0$$

$$x_0 \text{ is a zero of the polynomial } : \ P = (\lambda + 2)ax^2 + (\lambda + 1)bx + \lambda c \text{ and } x_0 \neq 0.$$

Let δ be the discriminant of P. We resume the general happening below:

If $a \neq 0$, reminding $P(x_0) = 0$, we get:

$\frac{\partial^2 f}{\partial x^2}(x_0, 0) = x_0^{\lambda-1} P'(x_0)$
$\frac{\partial^3 f}{\partial x^3}(x_0, 0) = x_0^{\lambda-2}(2(\lambda - 1)P'(x_0) + 2a(\lambda + 2)x_0).$

As $\lambda \neq 0, \ -1$ in the lists obtained concerning the whole reversible case above, different critical points have different critical values and $P'(x_0) = 0 \Leftrightarrow \delta = 0 \Leftrightarrow (\lambda + 1)b^2 = 4\lambda ac$, we have the following (remind that for the reversible case $b^2 - 4ac \neq 0$):

Lemma 3 *For $a \neq 0$ and $c \neq 0$, if $\delta \neq 0$ we obtain two different singular curves with a normal crossing, that is I_1 in Kodaira's classification and if $\delta = 0$, we obtain one singular fiber with a cusp,*

that is II. If $a \neq 0$ and $c = 0$, or $a = 0$ and $c \neq 0$, we obtain one singular fiber with normal crossing, that is I_1. Otherwise, there are no more singular fibers.

Finally, we are now able to compute all the singular fibers. The results are given in Tables 2, 3 and 4.

Fibration	$\{t=\infty\}$	$\{t=0\}$	t_1	t_2
(rv1)	$IV*$	I_2	I_1	I_1
(rv2)	$IV*$	I_2	I_1	I_1
(rv3)	IV	I_0*	I_1	I_1
(rv4)	IV	I_0*	I_1	I_1
(rv5)	$III*$	I_1	I_1	I_1
(rv6)	$III*$	I_1	I_1	I_1

Table 2: The elliptic reversible case with 4 singular fibers.

Fibration	$\{t=\infty\}$	$\{t=0\}$	t_1
(rv1) $a, b, c \neq 0,\ \delta = 0$	$IV*$	I_2	II
$a \neq 0$,$c = 0$	$III*$	I_2	I_1
$a = 0,\ c \neq 0$	$IV*$	III	I_1
(rv2) $a, b, c \neq 0,\ \delta = 0$	$IV*$	I_2	II
$a \neq 0$,$c = 0$	$III*$	I_2	I_1
$a = 0,\ c \neq 0$	$IV*$	III	I_1
(rv3) $a, b, c \neq 0,\ \delta = 0$	IV	I_0*	II
$a = 0$	IV	I_1*	I_1
(rv4), $a, b, c \neq 0,\ \delta = 0$	IV	I_0*	II
$a = 0$	IV	I_1*	I_1
(rv5) $a, b, c \neq 0,\ \delta = 0$	$III*$	I_1	II
$a \neq 0$,$c = 0$	$II*$	I_1	I_1
$a = 0,\ c \neq 0$	$III*$	II	I_1
(rv6) $a, b, c \neq 0,\ \delta = 0$	$III*$	I_1	II
$a \neq 0$,$c = 0$	$II*$	I_1	I_1
$a = 0,\ c \neq 0$	$III*$	II	I_1
(rv7)	$IV*$	III	I_1
(rv8)	$IV*$	III	I_1
(rv9)	I_1*	IV	I_1
(rv10)	I_1*	IV	I_1
(rv11)	III	$IV*$	I_1
(rv12)	III	$IV*$	I_1
(rv13)	$IV*$	III	I_1
(rv14)	$IV*$	III	I_1
(rv15)	$III*$	II	I_1
(rv16)	$III*$	II	I_1
(rv17)	$II*$	I_1	I_1.
(rv18)	$II*$	I_1	I_1.

Table 3: The elliptic reversible case with 3 singular fibers.

Fibration	$\{t=\infty\}$	$\{t=0\}$
$(i), k=1 \mod (4)$	III^*	III
$k=3 \mod (4)$	III	III^*
(ii) $k=1,2 \mod (6)$	II^*	II
$k=3,5 \mod (6)$	II	II^*

Table 4: The reversible case with 2 singular fibers.

3.2 The Lotka-Volterra case

3.2.1 Birational mappings

The same birational mapping: $X=\frac{1}{x}$ and $Y=\frac{y}{x}$, leads:

$$x^{\lambda}y^{\mu}(ax+by+c)=t$$

birationally to:

$$x^{-\lambda-\mu-1}y^{\mu}(cx+by+a)=t.$$

Using this, one can immediately verify that (lv1), (lv2), (lv3) and (lv5) are birationals and (lv4) is birational to (rlv3) and (rlv4). For the last Lotka-Volterra cases, there are another obvious birational mappings:

$$X=xy^{u}(1+y)^{v}\ ,\ Y=y$$

or:

$$Y=xy^{u}(1+y)^{v}\ ,\ X=x.$$

with $u,v\in\mathbb{Z}$ judiciously chosen.

3.2.2 The fibers $t=0$ and $t=\infty$

EXAMPLE 3:The singular fibers of (lv1)
Here the rational function is:

$$\frac{x^2y^3(z-x-y)}{z^6}.$$

The geometrical situation is explained in Figure 11. The intersection points with opposite multiplicity need to be blown-up. Here there are 3 points: $A_1=[0:1:0]$, $A_2=[1:0:0]$, $A_3=[1:-1:0]$.

Near A_1, locally the rational function becomes $\frac{X^2}{Z^6}$ such that we only need three blowing-ups to separate local branches. This situation is well-known like near A_2 and A_3, where we need respectively 2 and 6 blowing-ups. See Figures 12 and 13. The fiber at infinity is II^*. For the fiber $t=0$, we need two contractions as explained in Figure 14 and we finally get I_1.

EXAMPLE 4: The singular fibers of (vii) *with* $l=4 \bmod (6)$. Under birational equivalence, the rational function we need to consider is:

$$\frac{y^4(z+y)^5}{x^6z^3}.$$

We have to blow up 3 points: $A=[0:0:1]$, $B=[1:0:0]$ and $C=[0:1:-1]$. For A and C we have normal crossings and the situation is similar to precedent ones. We need to pay little more attention for the blowing-up of B:
Locally the rational function becomes: $\frac{Y^4(Y+Z)^5}{z^3}$. Here the first blowing- up separates the three branches. Now we need to blow-up the intersection point of the branch with multiplicity 6 and the branch with multiplicity -3. Locally the rational function is: $\frac{Y^6}{Z^3}$ and we get in a well-known situation. We obtain $II*$ for $t=0$. For $t=\infty$ we need three contractions to finally obtain II (see Figure 16).

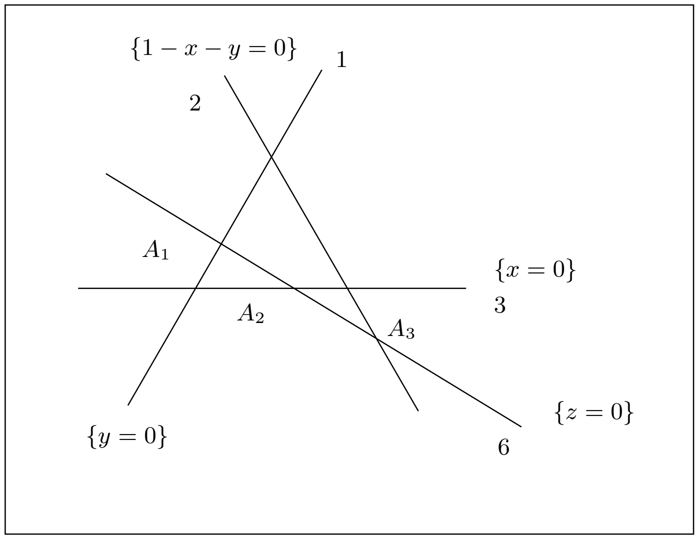

Figure 11: The geometrical situation of (lv1).

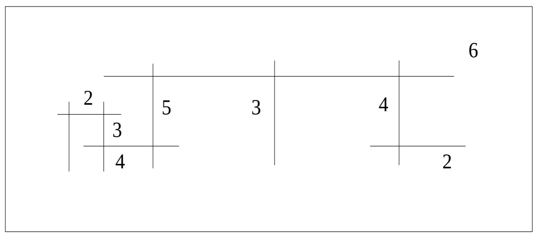

Figure 12: The fiber at infinity for (lv1).

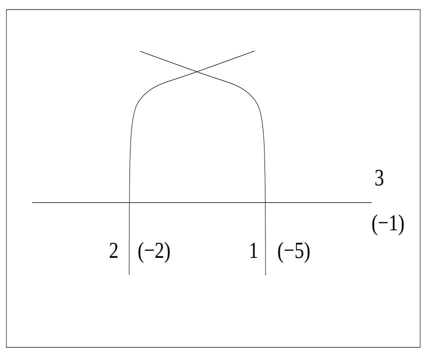

Figure 13: The fiber $t = 0$ for (lv1).

3.2.3 Other singular fiber

We write:

$$f = x^{\lambda} y^{\mu}(ax + by + c).$$

This is here elementary linear algebra and it immediately gives the following:

Lemma 4 *Under the assumptions $\lambda, \mu \neq 0, \lambda \neq -1, \mu \neq -1$ and $\lambda + \mu + 1 \neq 0$, the function above gives rise to another singular fiber if and only if $a, b, c \neq 0$ and the corresponding singular fiber is I_1.*

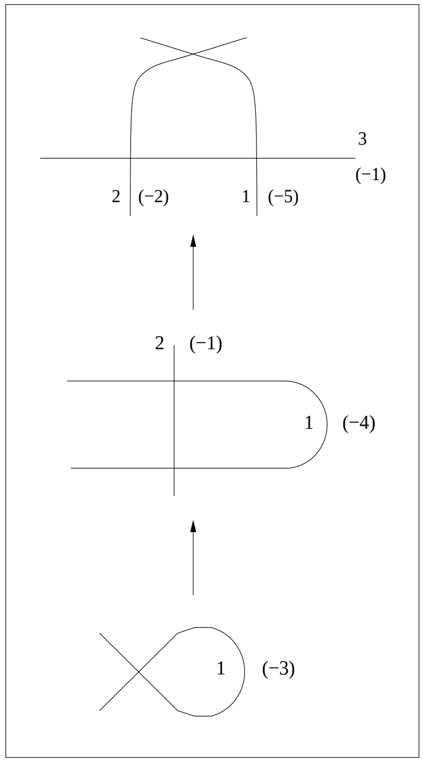

Figure 14: Contraction of the fiber $t = 0$ for (lv1).

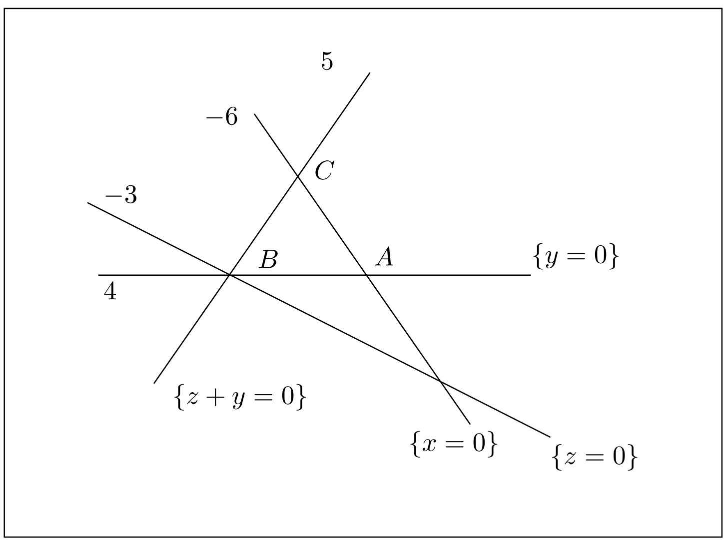

Figure 15: Geometrical situation of (vii).

Remark 2 *Such assumptions hold for our Lotka-Volterra and reversible Lotka-Volterra systems inducing elliptic fibrations.*

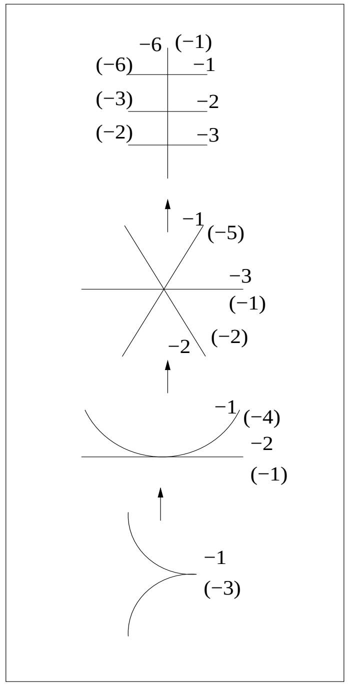

Figure 16: Contractions of $t = 0$ for (vii).

Fibration	Fiber $\{t = \infty\}$	Fiber $t = 0$	Other singular fiber
$(lv1)$	$II*$	I_1	I_1
$(lv2)$	$II*$	I_1	I_1
$(lv3)$	$II*$	I_1	I_1
$(lv5)$	$II*$	I_1	I_1
$(lv4)$	$III*$	I_2	I_1

Table 5: The Lotka-Volterra case with 3 singular fibers.

3.3 The reversible Lotka-Volterra case

The calculus are similar and are left to the reader. The results are contained in Tables 7 and 8.

Fibration	$\{t=\infty\}$	$t=0$
$(iii), k=l=1 \mod (3)$	IV^*	IV
$k=l=2 \mod (3)$	IV	IV^*
$(iv), k=l=1 \mod (3)$	IV^*	IV
$k=l=2 \mod (3)$	IV	IV^*
$(v), k=l=1 \mod (4)$	III^*	III
$k=l=3 \mod (4)$	III	III^*
$(vi), k=l=1 \mod (4)$	III^*	III
$k=l=3 \mod (4)$	III	III^*
$(vii), l=2 \mod (6)$	II^*	II
$l=4 \mod (6)$	II	II^*
$(viii), l=2 \mod (6)$	II^*	II
$l=4 \mod (6)$	II	II^*
$(ix), k=1 \mod (4)$	III^*	III
$k=3 \mod (4)$	III	III^*
$(x), k=1 \mod (4)$	III^*	III
$k=3 \mod (4)$	III	III^*
$(xi), k=1,2 \mod (6)$	II^*	II
$k=4,5 \mod (6)$	II	II^*
$(xii), k=1,2 \mod (6)$	II^*	II
$k=4,5 \mod (6)$	II	II^*

Table 6: The elliptic Lotka-Volterra case with 2 singular fibers.

Fibration	Fiber $\{t=\infty\}$	Fiber $t=0$	Other singular fiber
(rlv1)	$IV*$	I_3	I_1
(rlv2)	$IV*$	I_3	I_1
(rlv3)	$III*$	I_2	I_1
(rlv4)	$III*$	I_2	I_1
(rlv5)	IV	I_1*	I_1
(rlv6)	IV	I_1*	I_1

Table 7: The elliptic reversible Lotka-Volterra case with 3 singular fibers.

Fibration	Fiber $\{t=\infty\}$	Fiber $t=0$
$(ix), k=1 \mod (3)$	$IV*$	IV
$k=2 \mod (3)$	IV	$IV*$
$(x), k=1 \mod (3)$	$IV*$	IV
$k=2 \mod (3)$	IV	$IV*$
$(xi), k=1 \mod (4)$	$III*$	III
$k=3 \mod (4)$	III	$III*$
$(xii), k=1 \mod (4)$	$III*$	III
$k=3 \mod (4)$	III	$III*$

Table 8: The elliptic reversible Lotka-Volterra case with 2 singular fibers.

Acknowledgements:

The author would like to thank L. Gavrilov and I. D. Iliev and for their attention to the paper, many useful comments and stimulating remarks.

References

[BPV84] W. Barth , C. Peters, A. Van de Ven: *Compact complex surfaces*, Erg. Math., Springer-Verlag (1984).

[CGLPR] J. Chavarriga, B. Garciá, J. Llibre, J. S Pérez Del Río and J. A. Rodríguez: *Polynomial first integrals of quadratic vector fields*, Journal of Differential Equations Volume 230, Issue 2, 15 November 2006, Pages 393-421.

[CLLL06] G. Chen, C. Li, C. Liu, J. Llibre: *The cyclicity of period annuli of some classes of reversible quadratic systems*, Discrete Contin. Dyn. Syst. **16** (2006), no. 1, 157-177.

[G07] S. Gautier: *Feuilletages elliptiques quadratiques plans et leurs pertubations*, P.H.D Thesis, 7/12/2007.

[GGI] S. Gautier, L. Gavrilov, I. D. Iliev: *Pertubations of quadratic centers of genus one*, http://arxiv.org/abs/0705.1609.

[G01] L. Gavrilov: *The infinitesimal 16th Hilbert problem in the quadratic case*, Invent. math 143, 449-497 (2001).

[GH78] P. Griffiths and J. Harris: *Principles of algebraic geometry*, John Wiley and Sons, New York (1978).

[I98] I. D. Iliev: *Pertubations of quadratic centers*, Bull. Sci. math. 22 (1998). 107-161.

[ILY05] I. D. Iliev, C.Li, J.Yu: *Bifurcations of limit cycles from quadratic non-Hamiltonian systems with two centres and two unbounded heteroclinic loops*, Nonlinearity, 18 (2005), no. 1, 305–330.

[I02] Y. Ilyashenko: *Centennial history of Hilbert's 16th problem*, Amer. Math. So., 39 (2002), no 3, 301-354.

[J79] JC. Jouanolou: *Equations de Pfaff algébriques*, Lec. Notes in Math., 708 (1979).

[Ka75] K. Kas: *Weirstrass normal forms and invariants of elliptic surfaces*, Trans. of the A.M.S, 225, (1977).

[Ko60] K. Kodaira: *On compact analytic surfaces*, I, Annals of Math., 71 (1960),111-152.

[Ko63] K. Kodaira: *On compact analytic surfaces*, II, Annals of Math., 77 (1963),563-626.

[M89] R. Miranda, *The basic theory of elliptic surfaces*, Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research], ETS Editrice, Pisa, 1989.

[P90] G. S. Petrov: *Nonoscillation of elliptic integrals*, Funct. Anal. Appl., 3 (1990), 45-50

[YL02] J. Yu, C. Li: *Bifurcation of a class of planar non-Hamiltonian integrable systems with one center and one homoclinic loop*, J. Math. Anal. Appl., 269 (2002), no. 1, 227–243.